

PROBABILISTIC TECHNIQUES IN GEOTECHNICAL MODELLING – WHICH ONE SHOULD YOU USE?

W. S. Kaggwa¹ and Y.L. Kuo²

¹Senior Lecturer, ²EngTest Assistant, School of Civil, Environmental and Mining Engineering,
University of Adelaide, North Terrace, Adelaide, SA 5005, Australia.

ABSTRACT

Predictions of performance are at the core of geotechnical engineering design. Predictions based solely on deterministic analyses suffer from unquantifiable uncertainties and the implied absoluteness of the prediction. On the other hand, probabilistic estimates suffer from being vague, which is unsettling to most geotechnical engineers. Combining deterministic and probabilistic analyses offers synergies that are best utilized only if the geotechnical engineer appreciates the relative role of each type of analysis. This paper describes an overview of the hierarchy of probability-based analyses in geotechnical engineering predictions. The aim is to provide geotechnical engineers with a framework that integrates analytical and probabilistic analyses. The available probabilistic analyses, their level of complexity, applicability and limitations are considered, in order to enable the geotechnical engineer to choose correctly the optimum analysis that best suits their specific project and circumstances.

1 INTRODUCTION

There is increasing application of probabilistic methods and reliability analyses covering all aspects of geotechnical engineering, from site investigation planning to detail design. The complexities of the methods utilised and the effort involved, additional to traditional deterministic analyses, can be onerous. Duncan (2000) presented a method for incorporating reliability calculations in routine geotechnical engineering practice, and demonstrated the potential insight that probabilistic analysis offers the geotechnical engineer into the uncertainties inherent in their proposed design. Essentially, the variability of the prediction, whether serviceability-related or safety-related, is evaluated. This is then used to estimate the probability of occurrence of unsatisfactory conditions.

It is widely recognised that many geotechnical practitioners lack confidence in reliability-based analysis and design to substitute these for their conventional deterministic approaches (e.g. Focht and Focht, 2001). This is commonly attributed to unfamiliar jargon and methodology as well as the added effort necessary to implement probabilistic analyses. It is likely that the lack of a hierarchical approach and descriptions of alternative probabilistic approaches, also contribute to this state of confusion. Furthermore, there are other issues such as: the absence of mapping of different classes of probabilistic analyses and the associated complexity of methods of analysis and design; lack of appreciation of the links between extent of soil investigations and the evaluation of variability of site conditions and the absence of guidance on prediction method and the required complexity of prediction sought. All these issues tend to alienate many practitioners and clients. This leaves the practitioners and clients to rely on proven methods and techniques gained from experience using deterministic analyses, even in situations where probability-type analyses would be more appropriate. For example, the evaluation of the stability of a long highway embankment would be more meaningful when expressed in terms of the probability of failure, rather than the lowest factor of safety, because of the variability of ground conditions and the quality control during embankment construction.

Many geotechnical practitioners appreciate the need to undertake complex deterministic analyses where necessary. Simple models require simple tests, while sophisticated models necessitate sophisticated tests. Table 1, based on Jamiolkowski *et al.* (1985), clarifies the need to match the features of a chosen constitutive model to the determination of soil properties through soil testing. Kulhawy (2000) pointed out that although non-parallel links are possible, these really have to involve re-calibration and empiricism, say Category II model and Category III properties.

It is necessary to view and select probabilistic analyses within a comparable framework in order to match the complexity of the problem, the chosen analysis, and the sourcing of input statistical data. This paper addresses this aspect, including the development of combinations of analytical and probabilistic analyses.

2 PROBABILISTIC ANALYSES

Probabilistic analyses are indispensable when the geotechnical engineer wishes to explicitly evaluate the significance of variations of loads, ground conditions and soil properties, and to quantify the risks associated

with particular design options. Compared to loads, the variance of soil properties is considerably larger. Furthermore, the method chosen to analyse the field problem, being an idealisation of reality, introduces model uncertainty that often results in the prediction being different from field observation. Baecher and Christian (2003) described the commonly used probabilistic analyses for dealing with different sources of uncertainty as: first order second moment (FOSM), second-order-second-moment (SOSM), point estimation (PE) and Monte Carlo simulation (MCS).

Table 1: Categories of analytical methods for geotechnical modelling (after Jamiolkowski *et al.* 1985)

Category	Main feature of models	Determination of soil properties
I	Sophisticated constitutive models, including nonlinearity, elasto-plasticity, time and stress-dependency, and perhaps anisotropy	Numerous sophisticated laboratory tests, coupled with a range of <i>in situ</i> tests to assess <i>in situ</i> variables directly (such as stress)
II	Advanced constitutive models, including incremental elasto-plasticity or nonlinear elasticity	Conventional laboratory and <i>in situ</i> tests, coupled with a few special tests to define model limits, etc.
III	Simple linear models, such as isotropic elastic continuum (possibly layered) or empirical model	Conventional laboratory or <i>in situ</i> tests

With respect to soil statistical data, it is imperative to consider the key aspects of soil spatial variability, the implications of limited information, and local averaging. Furthermore, whether based on geotechnical site investigations or using results based on simulated random fields, two aspects of sampling require careful consideration. The first is the limited extent of sampling, in particular not performing enough measurements and therefore the impossibility of directly determining spatial variance and the scale of fluctuation.

The second limitation is related to scale effects because any type of measurement involves a finite volume of soil that differs from the volume used in modelling the soil mass. The volume of soil samples used in laboratory testing, or the disturbed soil zone in the case of field tests such as the vane shear and cone penetration test, is several orders of magnitude smaller than the volume of soil affected by a shallow or deep foundation. Accordingly, the variance determined from such measurements can be treated as providing point variance. A variance function is then necessary to relate how much the variance of a spatial average is reduced (due to averaging over the soil thickness L , and data correlation expressed by the scale of fluctuation, θ), when compared to the variance of the point values. Vanmarcke (1983) presented a number of variance functions including triangular, exponential, second-order autoregressive, and the squared exponential correlation functions. Based on the commonly used exponential form for the correlation function, the resulting variance function, $\gamma(L)$ is:

$$\gamma(L) = \frac{\sigma_L^2}{\sigma^2} = \frac{\theta^2}{2L^2} \left(\frac{2L}{\theta} - 1 + \exp\left(-\frac{2L}{\theta}\right) \right) \quad (1)$$

Where: L is the lag distance and θ is the scale of fluctuation.

The relationship between the variance function and the relative averaged size is shown in Figure 1. For a ratio (L/θ) of one order of magnitude (typical when the profile is divided into sub-layers), the appropriate variance is approximately one tenth of the point variance, whereas for two orders of magnitude (soil profile idealised as a single equivalent homogeneous layer), the variance is one percent of the point variance. Therefore, over-prediction of variance effects is unavoidable unless local average effects are explicitly considered.

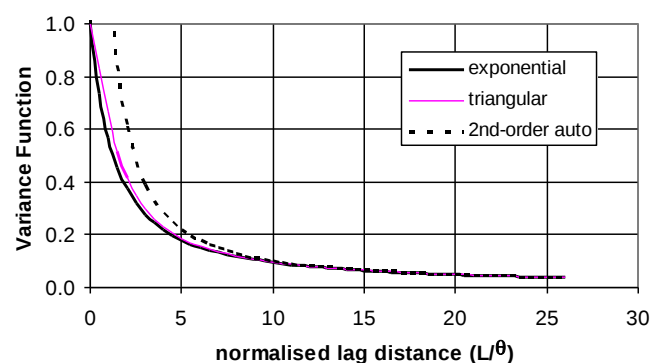


Figure 1: Variance reduction due to spatial averaging

3 CATEGORIES OF PROBABILISTIC ANALYSES

Using the categorisation of analytical methods described by Jamiolkowski *et al.* (1985), it is possible to group probabilistic analyses into three main categories, namely (i) simple methods that rely on statistical analyses, (ii) advanced analyses that account for the spatial, random variability of soil parameters within soil layers and (iii) sophisticated analyses that take into account the spatial variability of the soil parameters or soil response. The categories, their main features and the associated probability parameters are shown in Table 2.

Table 2: Proposed categories of probabilistic analyses

Category	Probabilistic analysis	Main features of probabilistic analysis and its implementation	Required probabilistic parameters
I	Sophisticated analysis	Sophisticated probabilistic analyses, including spatial variability of soil response (SVSR) or individual parameters (SVP) and their covariance, anisotropy, using MCS.	Upper and lower limits of soil non-linear response; mean, variance and scale of fluctuation; probability density function; spatial correlation structure.
II	Advanced analysis	Advanced probabilistic analyses that treat soil medium as spatially variable using spatial parameter variability (SPV), analysis captures second-order effects using SOSM, or MCS.	Mean and variance of variable soil parameters, probability density function, scale of fluctuation, and choice of spatial variance reduction function.
III	Basic analysis	Simple variance evaluation of homogeneous soil medium, using FOSM, PE, etc.	Statistical analysis of databases to obtain soil mean and variance, normal distribution function is adequate.

3.1 BASIC (CATEGORY III) PROBABILISTIC ANALYSES

In the simplest type of probabilistic analyses (Category III), the soil is idealised as a uniform homogeneous material and characterised by the mean and variance of each soil property and adoption of the normal distribution function. We can refer to such soil model as a single-element-model (SEM). The reduction of a soil profile to SEM requires the determination of weighted averages of the soil parameters. The effects of soil variance can be evaluated using standard statistical procedures, such as first order second moment (FOSM) techniques, and a normal probability distribution function is usually adequate. However, the choice of the variance to be used in Category III analyses is not a trivial matter. Where a lot of data is available, the variance of the mean (σ^2/N) should be used, where the point variance (σ^2) of the soil property is determined from N measurements. Alternatively, published data on variability may be used, such as those reported by Phoon and Kulhawy (1999) for many soil properties and parameters. However, these require correction (i) to ensure that only inherent soil variability is considered, and (ii) for variance reduction due to local averaging to be explicitly incorporated in the analysis. Results from different types of tests require correction to a standard test type with prior removal of systemic errors and variability due to the type of test. Separation of soil parameter variability into operator, test type and interpretation, and inherent variability involves collection of a lot of data and statistical analysis of the data.

3.2 ADVANCED (CATEGORY II) PROBABILISTIC ANALYSES

In Category II probabilistic analyses, the soil spatial variability is included explicitly. The probability distribution function, mean, standard deviation and scale of fluctuation, for each soil layer are required. Each soil layer is subdivided into elements (or sub-regions) with each element assigned a single value of the soil parameter in such a way as to satisfy the statistical parameters and the spatial correlation model. The Local Average Subdivision (LAS) algorithm described by Fenton and Vanmarcke (1990) has been used to generate accurate random fields based on point statistical moments, probability density function, and scale of fluctuation. It has been used in many theoretical studies of foundation, slope stability and seepage problems, using numerical methods and Monte Carlo simulation. The analyses can also be implemented in stochastic finite element (SFEM).

The subdivisions can be implemented in one, two or three dimensions. In one-dimensional modelling, vertical variability is modelled, neglecting any variations in the lateral directions, in conjunction with analytic solutions or finite layer analyses (e.g. Kaggwa *et al.* 2002). For two-dimensional problems (axi-symmetric or plane-strain), variations in one of the directions are neglected, together with stochastic finite element method (SFEM) or traditional finite element or finite difference methods and Monte Carlo simulation. For example, the stochastic finite element method was applied to pile settlement problems by Phoon *et al.* (1990) and Quek *et al.* (1992),

whereas Fenton and Griffiths (2002) and Griffiths *et al.* (2002) used MCS to the settlement of shallow foundations. In three-dimensional problems, variations in all spatial directions are included, for example Fenton and Griffiths (2005) and Goldsworthy *et al.* (2007).

3.3 SOPHISTICATED (CATEGORY I) PROBABILISTIC ANALYSES

Category I probabilistic analyses represent sophisticated techniques that combine statistical parameters and correlation structure. For implementation on a specific project, they require extensive and sophisticated testing of the soil in order to determine the correlation structure of soil parameters, including scale of fluctuation, auto-correlation function, or cross correlation functions among related soil properties. In complex problems including nonlinearity, soil yielding, and anisotropy, only sophisticated constitutive models are appropriate. Probabilistic analyses of these complex problems can be undertaken by considering the variability of the soil response and Monte Carlo simulation. This approach was proposed by Kaggwa (2000) and is briefly described.

3.4 EXAMPLE OF APPLICATION OF SPATIAL VARIABILITY OF SOIL RESPONSE (SVSR) IN SOPHISTICATED CONSTITUTIVE MODELS

Sophisticated constitutive models that combine nonlinearity, yielding, and anisotropy, involve many parameters, with two or more parameters being used to model a particular facet of soil behaviour. There is physical meaning attributable to some parameters (primary parameters) although other parameters (secondary parameters) are sometimes used for calibration of the constitutive model. More importantly, values of some parameters depend on the values of other soil parameters. If the parameters are treated as independent variables, there is potential for violation of the state boundary or failure criteria. Treating the model parameters as correlated variables adds further complexities because of the requirement when assigning values to the model parameters to ensure that there is no violation of the state boundary or failure criteria.

Soil variability is commonly treated in terms of individual parameter variability. For example linear elastic settlements are evaluated by considering the variability of soil compressibility, using Young's modulus of the soil, E , as the random variable. Since the variations in Young's modulus represent variations of the slope of the stress-strain relationship, what is modelled is the variability of the stress-strain soil response. Accordingly, this idea of modelling the variability of soil response can be extended to sophisticated constitutive models if a set of values of parameters is treated as the variable, rather than the individual soil parameters. Let us consider the simple linear elastic-plastic soil model defined by five parameters, namely stiffness parameters E' and ν' , shear strength parameters c' and ϕ' , and dilation angle φ . The set $R_i(E', \nu', c', \phi', \varphi)$ represents one combination of values of soil parameters that define the elasto-plastic soil response and different R_i represent different responses. When treated as a random variable, R_i can be used to model the spatially variable soil profile.

The overall procedure for a Monte Carlo simulation using finite element analysis is as follows.

1. Choose an appropriate constitutive model taking into account the complexity of field conditions, available soil data and the complexity of the required solution. It is possible to use different constitutive models for different materials, for example sand and clay layers, embankment fill and concrete.
2. For each soil layer or material, decide on the limits of the values of soil parameters. Check these limits in view of the common tendency to underestimate variability. Carry out numerical predictions using these bounds and check that the outputs bracket the range of expected behaviour.
3. Determine the most probable (expected) values of the model parameters and check that these values represent the median response. Again check that the output is as expected.
4. Decide on the number of sets n_s , covering the range of responses, preferably not less than seven and not greater than fifteen, unless there are good reasons to the contrary.
5. Determine the values of the constitutive model parameters for each set. A simple scalar can be used to assign values of primary parameters of the n_s sets. Check that each set satisfies the constitutive model and yields reasonable solutions when used in a deterministic analysis.
6. Using a normal probability distribution function and a coefficient of variation appropriate to the site conditions, generate a spatially variable finite element mesh, with each element assigned a material set number that falls within the range assigned to the soil layer. For Category II analysis, it is adequate for the response sets to be assigned to the elements as random numbers, whereas for Category I probabilistic analyses directly incorporate spatial correlation in the assignment of the response set to each element.
7. Carry out a deterministic prediction and obtain the desired outputs at the pre-selected locations.

8. Repeat steps 6 and 7 in a Monte Carlo simulation using a minimum of 2000 repetitions unless there are strong reasons to use fewer realisations.
9. Analyse the results of the Monte Carlo simulation to determine the mean and standard deviation of the output and the probability distribution function.
10. Make decisions about the response, based on the statistical information obtained from Step 9.

4 INTEGRATION OF ANALYTICAL AND PROBABILISTIC ANALYSES

In order for probabilistic techniques to be readily integrated in geotechnical engineering predictions, it is necessary to relate them directly to analytical methods and to view both as complimentary. The categories of probabilistic techniques, presented in Table 2, require linkage to the available methods of analysis presented in Table 1. There are many possible combinations of analytical and probabilistic methods, as shown in matrix form in Table 3. In extreme cases, the solution is obtained by putting all the effort in one type of analysis, either the deterministic-based analysis or database-based statistical analysis. This is particularly true of many numerical software programs that use sophisticated constitutive models, where a lot of effort is put into calibration of the numerical analysis, with little consideration of the statistical and spatial variability of the input parameters. Where parametric studies are undertaken, these are used to indicate the sensitivity of the solution to changes in parameter values.

The other extreme is the analysis of available data, using model tests or field measurements of say settlement of foundations on sand or liquefaction potential. Foundation geometry or earthquake-induced rock motion, combined with available field conditions obtained from some type of *in situ* or laboratory tests, are then used to develop charts, or equations, that characterise various levels of performance. Artificial neural network (ANN) applications, particularly Bayesian and fuzzy ANNs fall in the category of sophisticated probabilistic methods with no analytical treatment of the field problem.

The implementation of probabilistic analyses in complex geotechnical models has been hindered by a number of factors. These include many soil parameters in a constitutive model, inter-dependency of several parameters, difficulties in assigning correlations among model parameters and lack of feel for the geotechnical problem. Accordingly, most predictions that involve probabilistic methods fall in Categories II and III combined with analyses in Categories I and II.

Table 3: Methodologies that combine analytical methods and probabilistic analyses for geotechnical predictions

Category	<i>I</i> (sophisticated)	<i>II</i> (advanced)	<i>III</i> (basic)
<i>Category I (sophisticated)</i>	COMPLEX COMBINED	Analysis-intensive	Deterministic
<i>Category II (advanced)</i>	Probability- intensive	ADVANCED BALANCED	Analysis-driven
<i>Category III (simple)</i>	Probabilistic	Probability-driven	BASIC-SIMPLE

Semi-empirical methodologies use statistical methods to varying levels of complexity. These applications can be as simple as graphical representation of data and the determination of limits between acceptable, questionable and unacceptable design conditions. Regression analyses, where the scattered data is replaced by single equations have inherent limitations. The development of fuzzy artificial neural networks (ANNs) can be considered as examples of sophisticated probabilistic methods and empiricism.

The elements of the matrix in Table 3 represent methodologies where either the analytical method or probabilistic analysis drives the solution, as well as where complex analytical methods or probabilistic analyses are employed. Methodologies based on advanced analytical methods that employ simple probabilistic techniques are termed analysis-driven, whereas those that involve advanced probabilistic analyses but use simple analytical methods are termed probability-driven. Methodologies that rely on complex analytical methods and the use of advanced probabilistic techniques are termed analysis-intensive, whereas methodologies that rely on complex probabilistic analyses and the use of advanced analytical methods are termed probability-intensive. Those methodologies where analytical methods and probabilistic analyses of equal complexity are employed are represented by the diagonal elements in Table 3. These reflect equal complexity of the analytical method and probabilistic analysis and their features are summarised Table 4.

It can be seen from Table 4 that Complex-combined analyses are intensive in all respects. Because multiple outputs are sought, this necessitates sophisticated soil testing to obtain soil parameters and their variability that are then input into numerical analyses that involve complex constitutive models. Monte Carlo simulation is then utilised to obtain probabilistic data on the outputs. In Advanced-balanced methods, that would be appropriate for

most predictions where a single output is sought, an equation or straight-forward series of analysis steps are combined with simple probabilistic analyses to obtain information on the variance of the predicted response. For semi-empirical analyses, a line of best fit is obtained using least squares analysis and the residuals used to represent the scatter of the data from the expected behaviour or confidence limits.

Table 4: Hierarchy of probabilistic approaches and linkage to deterministic analyses

CLASS	FEATURES OF ANALYTICAL METHOD			FEATURES OF PROBABILISTIC TECHNIQUE	
	Soil model	Parameter evaluation	Output from analysis	Soil variability	Evaluation of variability
I (Complex-combined)	Quite sophisticated constitutive models, including nonlinearity, elasto-plasticity, time and stress-dependency, and perhaps anisotropy e.g. Cam Clay, stress-dependent elasto-plasticity	Multiple parameters, some parameters depend on others. Numerous sophisticated laboratory tests, coupled with a range of <i>in situ</i> tests to assess <i>in situ</i> variables directly (e.g. stress)	Multiple outputs required, e.g. Pore pressure, vertical settlement, horizontal deflections, time effects for embankment and deep excavations.	Spatial variability within soil layers, correlation structure, anisotropy, and MSC methodology for solution	Exhaustive evaluation of statistical parameters, probability density function, anisotropic correlation structure
II (Advanced-balanced)	Advanced constitutive models, including incremental elasto-plasticity or nonlinear elasticity	A few key parameters. Conventional laboratory and <i>in situ</i> tests, coupled with a few special tests to define model limits, etc	Single response, e.g. vertical settlement of footing, bearing capacity of footing	Spatial variability, variance reduction to account for correlation structure	Conventional statistical analysis of soil data
III (Basic-simple)	Simple linear models, such as isotropic elastic continuum (or layered).	One or two average parameters. Conventional laboratory or <i>in situ</i> tests	Single response, e.g. elastic settlement of footing	Averaged value of parameter over soil profile, or characteristic value	Conventional statistical analysis of soil data to account for variance of averages

5 EXAMPLE – PROBABILISTIC BEARING CAPACITY OF STRIP FOOTING ON PURELY COHESIVE SOIL

The Prandtl (1921) solution of bearing capacity factor N_c of a strip footing, width B , on the surface of a homogeneous purely cohesive soil with cohesion c , is widely used in geotechnical engineering. Because it is underpinned by theory, it can be considered to fall in Category II analytical method. The solution can be written as (Meyerhof, 1963):

$$q_u = (2 + \pi)c = cN_c \quad (2)$$

In order to account for non-homogeneity, the soil can be modelled as a variable material with the level of complexity depending on the importance of the structure and the accuracy of the estimate being sought. Referring to Table 2, Category III analyses would involve statistical treatment of the variations of the averaged value of c for the homogeneous soil layer. Adoption of FOSM solution technique can show that the mean value of bearing capacity μ_{q_u} is given by:

$$\mu_{q_u} = \mu_c N_c \quad (3)$$

The standard deviation of bearing capacity σ_{q_u} is given by:

$$\sigma_{q_u} = \sigma_c \quad (4)$$

With respect to *Category II* in Table 2, evaluation of the mean and standard deviation of the bearing capacity requires consideration of the spatial variability of c_u , and involves subdivision of the soil medium into sub-layers or elements. A first approximation may assume an uncorrelated structure, in which case the values of c assigned to each element would only take into account the mean, standard deviation, and probability density function

appropriate to the soil layer. Analyses using Monte Carlo simulation of the shear strength values and numerical solution of the FEM problem would then be undertaken. Table 5 shows a summary of the results of probabilistic studies.

Table 5: Summary of results of probabilistic studies

Category	Method used in analysis	Mean of N_c (μ_{N_c}/μ_c)	Variance of N_c (COV_{N_c}/COV_c)
I	Random field simulation with correlation length (θ), analysis using MCS	Decreases with increasing COV_c but $COV_{N_c} \approx 1$ for COV_c less than 20%	Tends towards 0.1 for low values of θ/B Tends towards 1 for large θ/B
II	Random spatial variability, analysis using MCS	1	Variable, depends on relative size of adopted elements
III	First Order Second Moment (single layer)	1	1

For *Categories I* analyses, this requires considerations of the spatial variability (by selecting an appropriate value of the correlation length θ) in order to simulate the correlation structure of the random values. That is, the mean μ_c , coefficient of variation COV_c , correlation length θ/B and probability density function are required inputs.

There are very few instances reported in the literature where complex numerical analyses have been combined with complex probabilistic applications. One example of such study was undertaken by Kuo (2008). The *Category I* analyses of a strip footing at the surface of a cohesive soil layer used stochastic random field and finite element limit analysis of lower and upper bound bearing capacity. Kuo (2008) adopted a log-normal probability density function and exponential variance reduction function to simulate random field. The results highlight the impact of complexity of probabilistic analysis on bearing capacity factor and are used here as a benchmark for the lower-category analyses. The significance of soil variability with respect to the expected value, and the variance, of the bearing capacity factor are shown in Figures 2 and 3. The common assumption of constant expected value is reasonable as long as the COV of cohesion is less than 20%.

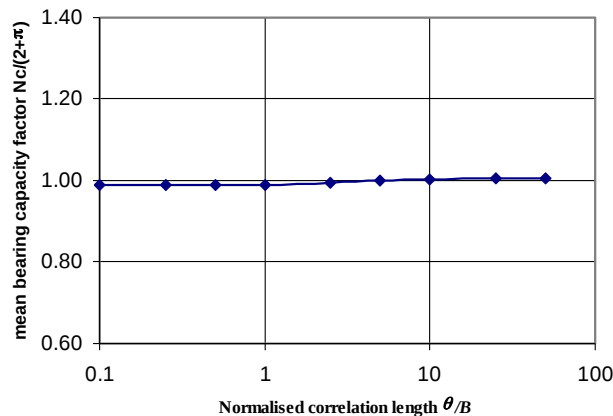


Figure 2: Effect of correlation length on expected value of bearing capacity factor N_c

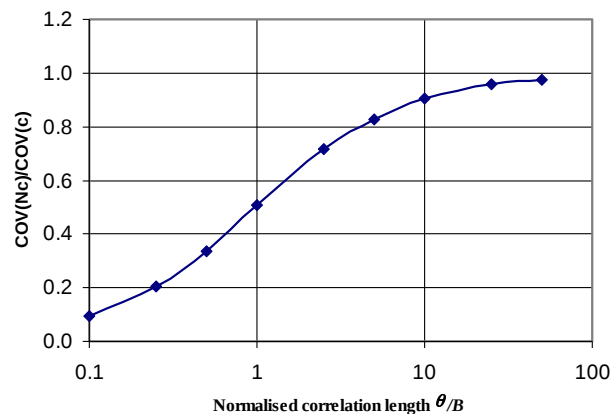


Figure 3: Effect of correlation length on variability of bearing capacity factor N_c

However, the correlation length has a major effect on the variability of the bearing capacity factor as shown in Figure 3. Low values of correlation length (which imply random variations over short distances) result in low variability of the bearing capacity factor and can be explained by the variance reduction effects. For a given strip footing width, the small correlation length compared to its width results in a reduction of the variance. For large correlation lengths relative to the footing width, the variance tends to a homogeneous, single variance problem.

6 SUMMARY AND CONCLUSIONS

The application of probabilistic analyses in routine geotechnical engineering predictions requires careful consideration of the methodology to follow, in particular linking the analytical method and the probabilistic technique to be employed. The choices available have been described, highlighting potential biases towards either analytical method or probabilistic technique. These have the potential to offer the engineer a strategy for checking the errors inherent in the simpler categories of methodologies.

7 REFERENCES

- Baecher, G.B. and Christian, J.T.(2003). Reliability and statistics in geotechnical engineering. John Wiley & Sons Ltd. Chichester, England.
- Duncan, J.M. (2000). Factor of safety and reliability in geotechnical engineering. *J. Geotech. Geoenviron. Eng.*, ASCE, 126(4), 307-316.
- Fenton, G.A. and Vanmarcke, E.H. (1990). Simulation of random fields via local average subdivision. *J. Engr. Mech.* 116(8), 1733-1749.
- Fenton, G.A. and Griffiths, D.V. (2002). Probabilistic foundation settlement on spatially random soil. *J. Geotech. Geoenviron. Eng.*, ASCE, 128(5), 381-390.
- Fenton, G.A. and Griffiths, D.V. (2005). Three-dimensional probabilistic foundation settlement. *J. Geotech. Geoenviron. Eng.*, ASCE, 131(2), 232-239.
- Focht Jr., J.A. and Focht III, J.A. (2001). Factor of safety and reliability in geotechnical engineering. *J. Geotech. and Geoenviron. Eng.*, ASCE, 127(8), 704-706.
- Goldsworthy, J.S., Jaksa, M.B., Fenton, G.A., Kaggwa, W.S., Griffiths, D.V. and Poulos, H.G. (2007). Effect of sample location on the reliability based design of pad foundations. *Georisk.* 1(3). 155-166.
- Griffiths, D.V., Fenton, G.A. and Manoharan, N. (2002). Bearing capacity of rough strip footing on cohesive soil: Probabilistic study. *J. Geotech. Geoenviron. Engrg.*, ASCE, 128(9), 743-755.
- Jamiolkowski, M., Ladd, C.C., Germaine, J.T., and Lancelotta, R. (1985). New developments in field and laboratory testing of soils. *Proc. 11th Intl. Conf. on Soil Mechanics & Foundation Engrn*1. (1), San Francisco: 57-154.
- Kaggwa, W.S. (2000). Determination of the spatial variability of soil design parameters and its significance in geotechnical engineering analyses. *J.R. Booker Mem. Symp. Developments in Theoretical Geomechanics*. D. Smith & J.P. Carter (eds). 681-698.
- Kaggwa, W.S., Cheong, M.T. and Jaksa, M.B. (2002). Assessment of the luck associated with the settlement predictions that are based on elastic theory. *Proc.Intl. Conf. on Probabilistics in GeoTechnics: Technical and Economic Risk Estimation*. R. Pottler, H Klapperich & H. Schweiger (eds). Graz, Austria. 79-86.
- Kulhawy, F. H. (2000). Some issues in geotechnical engineering. *J.R. Booker Mem. Symp. Developments in Theoretical Geomechanics*. D. Smith & J.P. Carter (eds). pp. 677-680.
- Kuo, Y.K. (2008). Effect of soil variability on the bearing capacity of footings on multi-layered soil. PhD Thesis, University of Adelaide. 350p.
- Meyerhof, G.G. (1963). Some recent research on the bearing capacity of foundations. *Canadian Geotech Journ.* 1(1), 16-26.
- Phoon, K.K. and Kulhawy, F.H. (1999). Evaluation of geotechnical property variability. *Can. Geotech. J.* 36(4) 625-639.
- Phoon, K.K., Quek, S.T., Chow, Y.K. and Lee, S.L. (1990). Reliability analysis of pile settlement. *J. Geotech. Engrg.*, ASCE, 116(11), 1717-1735.
- Prandtl, L. (1921). Über die eindringungs-festigkeit (Harte) plastischer bautoffe und die festigkeit von schneiden. *Zeitschrift für Angewandte Mathematik und Mechnik.* 1, 15–20.
- Quek, S.T., Chow, Y.K. and Phoon, K.K. (1992). Further contributions to reliability-based pile-settlement analysis. *J. Geotech. Engrg.*, ASCE, 118(5), 726-742.
- Vanmarcke, E. (1983). Random fields: analysis and synthesis, The MIT Press, Cambridge, Mass.